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Sharp bounds on the number of resonances for conformally compact manifolds with constant negative curvature near infinity

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Let (X, g) be a complete, conformally compact, n-dimensional Riemannian manifold, $n \geq 2$, with constant negative curvature (which may be supposed to be -1) near infinity. The metric g is of the form $g = \rho^{-2}h$, where $\rho \in C^\infty(\overline{X})$, $\rho|_{\partial X} = 0$, $d\rho|_{\partial X} \neq 0$, $\rho > 0$ in X, h is a Riemannian metric on X of class $C^\infty(\overline{X})$. Denote by Δ_X the Laplace-Beltrami operator on (X, g) and define the resolvent

$$R(s) = (\Delta_X - s(n-1-s))^{-1} : L^2(X) \to L^2(X), \quad \mathrm{Re}\, s \gg 1,$$

where $L^2(X) = L^2(X, d\mathrm{Vol}_g)$. Then,

$$R(s) : L^2_{comp}(X) \to L^2_{loc}(X)$$

extends meromorphically to the whole complex plane $\mathbf{C}$. This fact was proved by Mazzeo-Melrose [7] for a larger class of manifolds (see also [2]). The poles of this continuation are called resonances and the multiplicity of a resonance $s_0 \in \mathbf{C}$ is defined as the rank of the operator

$$\int_{\gamma(s_0)} (n-1-2s)R(s)ds,$$

where $\gamma(s_0)$ is a circle centered at s_0 containing no other poles. Denote by $\mathcal{R}_X$ the set of all resonances repeated according to the multiplicity, and set

$$N_X(r) = \#\{s \in \mathcal{R}_X : |s| \leq r\}, \quad r > 1.$$

Guillopé and Zworski [2] proved that $N_X(r) = O(r^{n+1})$. Moreover, in the case of $n = 2$ they obtained a better bound $N_X(r) = O(r^2)$ (see [3]) as well as a lower bound $N_X(r) \geq r^2/C$, $C > 0$, under a natural assumption (see [4]). Our main result is the following

Theorem 1. *For any conformally compact manifold (X, g) as above, the following upper bound holds:*

$$N_X(r) \leq Cr^n \tag{1}$$

with a constant $C > 0$.

Note that such a bound is proved by Patterson-Perry [10] for a class of quotions $\Gamma \setminus \mathbf{H}^n$ with n even via the properties of the dynamical zeta function. Perry [11] has recently obtained sharp lower bounds of the form $N_X(r) \geq r^n/C$ for such quotions.

Sharp upper bounds on the number of resonances have been obtained in the case of Euclidean scattering. Melrose [8] first obtained a bound of the form (1) for obstacle scattering in odd dimensions. Later on he used this bound in an essential way to prove the Weyl asymptotic for the scattering phase in this case (see [9]). Zworski [20] obtained such a sharp upper bound for potential scattering in odd dimensions, as well as an asymptotic of the number of resonances for a class of radial potentials (see [19]). Vodev [14] proved a sharp upper bound like (1) for metric perturbations of the Lapalcian and extended this result to more general compactly supported perturbations not necessarily self-adjoint and elliptic still in odd dimensions (see [15]). He also obtained sharp upper bounds on the number of resonances in even dimensions (see [18]). Sjöstrand and Zworski [13] proved sharp upper bounds on the number of resonances for a large class of self-adjoint compactly supported perturbations in odd dimensions using the complex scaling method. In the case of semi-classical problems Sjöstrand [12] obtained a semi-classical analogue of the bound (1).

The bound (1) follow from the following upper bounds.

Proposition 2. *For $\forall\, 0 < \varepsilon \ll 1\ \exists C_\varepsilon > 0$ so that*

$$\sharp\{s \in \mathcal{R}_X : r/2 \leq |s| \leq r, s \in \mathbf{C}_\varepsilon\} \leq C_\varepsilon r^n, \quad r > 1, \tag{2}$$

where $\mathbf{C}_\varepsilon := \mathbf{C} \setminus \{s \in \mathbf{C} : \pi - \varepsilon \leq \arg s \leq \pi + \varepsilon\}$.

Proposition 3. *For $\forall\, 0 < \varepsilon \ll 1\ \exists \widetilde{C}_\varepsilon > 0$ so that*

$$\sharp\{s \in \mathcal{R}_X : |s| \leq r, s \in \widetilde{\mathbf{C}}_\varepsilon\} \leq \widetilde{C}_\varepsilon r^n, \quad r > 1, \tag{3}$$

where $\widetilde{\mathbf{C}}_\varepsilon := \{s \in \mathbf{C} : \pi/2 + \varepsilon \leq \arg s \leq 3\pi/2 - \varepsilon\}$.

Note that the bound (3) has been announced in [5] where a short sketch of the proof is presented.

Idea of proof of Proposition 2. To prove (2) we modify the parametrix for the resolvent constructed by Guillopé-Zworski [2] (who followed the more general construction of Mazzeo-Melrose [7]). Denote

$$(\mathbf{H}^n, g_0) := (\mathbf{R}_x^{n-1} \times \mathbf{R}_y^+,\ y^{-2}(dx^2 + dy^2))$$

with Laplace-Beltrami operator given by

$$\Delta_{\mathbf{H}^n} = -y^2\partial_y^2 + (n-2)y\partial_y + y^2\Delta_x, \quad \Delta_x = -\sum_{j=1}^{n-1}\partial_{x_j}^2.$$

Denote $L^2(\mathbf{H}^n) := L^2(\mathbf{H}^n; d\mathrm{Vol}_{g_0})$. Following [2], given any integer $N \gg 1$, we construct operators

$$\mathcal{F}_N(s) : y^N L^2(\mathbf{H}^n) \to y^{-N}H^2(\mathbf{H}^n), \quad \mathcal{P}_N(s) : y^N L^2(\mathbf{H}^n) \to y^N L^2(\mathbf{H}^n),$$

defined for $\mathrm{Re}\, s > -N + (n-1)/2$ and depending meromorphically on s with poles at $-k$, $k \in \mathbf{N}$, so that

$$(\Delta_{\mathbf{H}^n} - s(n-1-s))\mathcal{F}_N(s) = \chi_0 + \mathcal{P}_N(s), \tag{4}$$

where $\chi_0 = \varphi(x)\psi(y)$, $\varphi \in C_0^\infty(\mathbf{R}^{n-1})$, $\psi \in C^\infty(\mathbf{R})$, $\psi(y) = 1$ for $y \leq 2\delta_0$, $\psi(y) = 0$ for $y \geq 3\delta_0$, $0 < \delta_0 \ll 1$. Moreover, the operator $y^{-N}\mathcal{P}_N(s)y^N$ is trace class on $L^2(\mathbf{H}^n)$. Given a compact operator A, denote by $\mu_k(A)$ its characteristic values, i.e. the eigenvalues of $(A^*A)^{1/2}$.

Lemma 4. *There exists $0 < \gamma_0 < 1$ (independent of s and N) so that if δ_0 is taken small enough (independent of s and N), for $s \in \mathbf{C}_\varepsilon$, $|s| \leq \gamma_0 N$, we have*

$$\|y^{-N}\mathcal{P}_N(s)y^N\|_{\mathcal{L}(L^2(\mathbf{H}^n))} \leq e^{C_1 N}, \tag{5}$$

$$\mu_k\left(y^{-N}\mathcal{P}_N(s)y^N\right) \leq e^{-C_2 N}k^{-2} \quad for \quad k \geq C_3 N^{n-1} \tag{6}$$

with constants $C_1, C_2, C_3 > 0$ independent of s, N and k. Moreover, for $s \in \mathbf{C}_\varepsilon$, $|s| \leq \gamma_0 N$, $\mathrm{Re}\, s \geq \gamma_0 N/2$, we have

$$\|y^{-N}\mathcal{P}_N(s)y^N\|_{\mathcal{L}(L^2(\mathbf{H}^n))} \leq e^{-C_4 N}, \tag{7}$$

with a constant $C_4 > 0$ independent of N and s.

It is shown in [2], Lemma 3.1, that there exists a neighbourhood Y of ∂X in $\overline{X}$ and an open covering $Y \subset \cup_{j=1}^M Y_j$ such that each Y_j is isometric to $U = \{(x,y) \in \mathbf{H}^n : |x|^2 + y^2 < 1\}$. Following [2] we denote by ι_j the isometry from Y_j to U, and by ι_j^* the induced pull-back operation transforming operators acting on functions in U to operators acting on functions in Y_j. We have $\rho \circ \iota_j^{-1} = y + O(y^2)$. Furthermore, there exists a partition of the unity in X, $\{\chi^j\}$, $\mathrm{supp}\,\chi^j \subset Y_j$, of the form $\chi^j = \varphi^j\psi^j$ with $\varphi^j \in C^\infty(\partial X)$, $\mathrm{supp}\,\varphi^j \subset \overline{Y}_j \cap \partial X$, $\sum_{j=1}^M \varphi^j = 1$ so that $\varphi^j \circ \iota_j^{-1}$ depends only on the variable x and $\psi^j \circ \iota_j^{-1}$ depends only on the variable y. Moreover, taking Y properly one can arrange that $\psi^j \circ \iota_j^{-1} = 1$ for $y \leq \delta$, $\psi^j \circ \iota_j^{-1} = 0$ for $y \geq 2\delta$ with some $0 < \delta \ll 1$ independent of j. Thus the function $\chi = \sum_{j=1}^M \chi^j$ is equal to 1 in $\{\rho \leq \delta\}$ and to zero in $\{\rho \geq 2\delta\}$. It is clear that to each function $\chi^j \circ \iota_j^{-1} \in C^\infty(\overline{U})$ we can associate operators $\mathcal{F}_N^j(s)$ and $\mathcal{P}_N^j(s)$ satisfying (4) with χ_0 replaced by $\chi^j \circ \iota_j^{-1}$. Setting

$$F_N(s) = \sum_{j=1}^M \iota_j^*\mathcal{F}_N^j(s)\iota_j^{*-1}, \quad P_N(s) = \sum_{j=1}^M \iota_j^*\mathcal{P}_N^j(s)\iota_j^{*-1},$$

we have

$$(\Delta_X - s(n-1-s))F_N(s) = \chi + P_N(s). \tag{8}$$

Moreover, for $s \in \mathbf{C}_\varepsilon$, $\operatorname{Re} s > -N + (n-1)/2$, the operator $\rho^N F_N(s)\rho^N$ is bounded on $L^2(X)$, while $\rho^{-N} P_N(s)\rho^N$ is a trace class operator on $L^2(X)$ and, in view of Lemma 4, satisfies an analogue of the bounds (5)-(7) with possibly new constants.

Let $\eta \in C_0^\infty(X)$, $\eta = 0$ in $\{\rho \le \delta/2\}$, $\chi = 1$ on $\operatorname{supp}(1-\eta)$, and let $s_N = 2\gamma_0 N/3$. Using (8) one can easily get

$$\rho^N R(s)\rho^N \left(I - K_N(s, s_N)\right) = \widetilde{K}_N(s, s_N), \tag{9}$$

where

$$\begin{aligned} K_N(s, s_N) &= -\rho^{-N} P_N(s)\rho^N - \rho^{-N}[\Delta_X, \eta]R(s_N)(1-\chi)\rho^N \\ &\quad -(s_N(n-1-s_N) - s(n-1-s))\rho^{-N}\eta R(s_N)(1-\chi)\rho^N, \\ \widetilde{K}_N(s, s_N) &= \rho^N \eta R(s_N)(1-\chi)\rho^N + \rho^N F_N(s)\rho^N. \end{aligned}$$

The operator $K_N(s, s_N)$ is analytic in $\{s \in \mathbf{C}_\varepsilon, \operatorname{Re} s > -N+(n-1)/2\}$ with values in the compact operators on $L^2(X)$ and the operator $\widetilde{K}_N(s, s_N)$ is analytic in $\{s \in \mathbf{C}_\varepsilon, \operatorname{Re} s > -N + (n-1)/2\}$ with values in the bounded operators on $L^2(X)$. Moreover, in view of (7), we have

$$\|K_N(s_N, s_N)\|_{\mathcal{L}(L^2(X))} \le 1/2.$$

Now it follows from (9) and the appendix in [18] that the poles of $\rho^N R(s)\rho^N$ in $\{s \in \mathbf{C}_\varepsilon, \operatorname{Re} s > -N + (n-1)/2\}$ are among (with multiplicities) the zeros of the function

$$h_N(s) = \det\left(I - (K_N(s, s_N)^{n+3} - K_N(s_N, s_N)^{n+3})(I - K_N(s_N, s_N)^{n+3})^{-1}\right)$$

which is well defined and analytic in this region, and $h_N(s_N) = 1$. Thus, the bound (2) follows from Carleman's theorem (e.g. see [6]) and the following

Lemma 5. *For $s \in \mathbf{C}_\varepsilon$, $|s| \le \gamma_0 N$, we have*

$$|h_N(s)| \le \begin{cases} e^{CN^n}, & \\ e^{C(|s-s_N|+1)^n} & \text{if} \quad \operatorname{Re} s \ge \gamma_0 N/2, \end{cases} \tag{10}$$

with a constant $C > 0$ independent of s and N.

Idea of proof of Proposition 3. It consists of using the properties of the scattering operator $S(s) : C^\infty(\partial X) \to C^\infty(\partial X)$. Recall that the Schwartz kernel of $S(s)$ is defined by

$$S(s)(m_\infty, m'_\infty; s) = (2s - n + 1) \lim_{m \to m_\infty} \lim_{m' \to m'_\infty} \rho(m)^{-s} \rho(m')^{-s} R(s)(m, m'),$$

where $m_\infty, m'_\infty \in \partial X$. One can show that $S(s)$ is a meromorphic family with poles coinciding with the resonances and the multiplicities agree. Moreover, we have

$$S(s)S(n-1-s) = I, \tag{11}$$

$$S(s) = c(s)\Delta_{\partial X}^{s-(n-1)/2} + \text{smoothing operator},$$

where $\Delta_{\partial X}$ is the Laplace-Beltrami operator on $(\partial X, \partial h)$, ∂h being the Riemannian metric on ∂X induced by the metric h, and

$$c(s) = 2^{-2s+n-1} \frac{\Gamma(-s + (n-1)/2)}{\Gamma(s - (n-1)/2)}.$$

More precisely,

$$c(n-1-s)(P_0+\Delta_{\partial X})^{(n-1)/2-s}S(s)=I+K(s), \tag{12}$$

where P_0 denotes the orthogonal projection on $\operatorname{Ker}\Delta_{\partial X}$, and $K(s)$ is analytic in $\operatorname{Re} s \geq \gamma \gg 1$ with values in the trace class operators on $L^2(\partial X)$. Thus the function

$$h(s)=\det(I+K(s))$$

is well defined and analytic in $\operatorname{Re} s \geq \gamma$. By (11) and (12) we conclude that the poles of $R(s)$ in $\operatorname{Re} s \leq n-1-\gamma$, with $\gamma \gg 1$, are among the poles of $(I+K(n-1-s))^{-1}$, and hence, in view of the Proposition in the appendix of [18], among the zeros (with multiplicity) of the function $h(n-1-s)$ in $\operatorname{Re} s \leq n-1-\gamma$. Thus, the bound (3) follows from Carleman's theorem and the following

Lemma 6. *For* $\operatorname{Re} s \geq \gamma \gg 1$, *we have*

$$|h(s)| \leq e^{C|s|^n} \tag{13}$$

with a constant $C>0$ *independent of* s.

References

[1] L. Guillopé, *Fonctions zeta de Selberg et surfaces de géométrie finie*, Advanced Studies in Pure Math. **21** (1992), 33-70.

[2] L. Guillopé and M. Zworski, *Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity*, Asympt. Anal. **11** (1995), 1-22.

[3] L. Guillopé and M. Zworski, *Upper bounds on the number of resonances for non-compact Riemann surfaces*, J. Funct. Anal. **129** (1995), 364-389.

[4] L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. Math. **145** (1997), 597-660.

[5] R. G. Froese and P. D. Hislop, *On the distribution of resonances for some asymptotically hyperbolic manifolds*, Journées *Equations aux dérivées partielles*, Nantes, 2000.

[6] B. Ja. Levin, *Distribution of zeros of entire functions*, Trans. Math. Monographs **5**, Amer. Math. Soc., Providence, 1964.

[7] R. Mazzeo and R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, J. Funct. Anal. **75** (1987), 260-310.

[8] R. Melrose, *Polynomial bounds on the distribution of poles in scattering by obstacle*, in: "Journées Equations aux dérivées partielles", Saint-Jean des Monts, 1984.

[9] R. Melrose, *Weyl asymptotic for the phase in obstacle scattering*, Commun. Partial Diff. Equations **13** (1988), 1431-1439.

[10] S. J. Patterson and P. A. Perry, *The divisor of Selberg's zeta function for Kleinian groups*, Duke Math. J. **106** (2001), 321-390.

[11] P. A. Perry, *Poisson formula and lower bounds on resonances for hyperbolic manifolds*, J. Funct. Anal., to appear.

[12] J. SJÖSTRAND, *Geometric bounds on the density of resonances for semi-classical problems*, Duke Math. J. **60** (1990), 1-57.

[13] J. SJÖSTRAND AND M. ZWORSKI, *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc. **4** (1991), 729-769.

[14] G. VODEV, *Sharp polynomial bounds on the number of scattering poles for perturbations of the Laplacian in* $\mathbf{R}^n$, Math. Ann. **291** (1991), 39-49.

[15] G. VODEV, *Sharp bounds on the number of scattering poles for perturbations of the Laplacian*, Commun. Math. Phys. **146** (1992), 205-216.

[16] G. VODEV, *On the distribution of scattering poles for perturbations of the Laplacian*, Ann. Inst. Fourier, Grenoble, **42** (1992), 625-635.

[17] G. VODEV, *Sharp bounds on the number of resonances for symmetric systems*, Ann. Inst. Henri Poincaré (Physique Théorique) **62** (1995), 401-407.

[18] G. VODEV, *Sharp bounds on the number of scattering poles in even-dimensional spaces*, Duke Math. J. **74** (1994), 1-17.

[19] M. ZWORSKI, *Sharp polynomial bounds on the number of scattering poles of radial potentials*, J. Funct. Anal. **82** (1989), 370-403.

[20] M. ZWORSKI, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. **59** (1989), 311-323.